Subject DISTORTION EFFECTS IN BEAM TRANSPORT SYSTEMS WITH SPACE CHARGE

## I. Introduction

Several authors[1] have expressed concern over the distortions introduced by the fringing fields in the quadrupoles of a transport system. This note is an effort to identify the relevant parameters and to determine the seriousness of the effect. Space charge is included in a linear approximation.

## II. Transport System

We shall study the quadrupole fringing fields in a transport system which is taken to be periodic and non-relativistic in order to simplify the analysis. The system will consist of thin lenses (triplets) of focal length f separated by a length L. This permits us to retain circular symmetry. It is further assumed that the beam is matched so that a "waist" is reached at the center of each lens and midway between lenses.

The envelope equation for an unbunched beam is

$$a'' + Ka = \frac{P^2}{a^3} + \frac{Q}{a} \tag{1}$$

where K is proportional to the external focusing force, $\pi P$ is the x-x' phase space area, and

$$Q = \frac{60\ (\text{ohms})\ e I\ (\text{amp})}{Mc^2 \beta^3}, \tag{2}$$

is the space charge parameter . At a thin lens of focal length f , we have

$$\frac{\Delta a'}{a} = -\frac{1}{f} , \tag{3}$$

and in free space

$$a'' = \frac{P^2}{a^3} + \frac{Q}{a} . \tag{4}$$

Assuming a waist midway between lenses, which is taken as the origin of the coordinate system, one can integrate Eq.(4) to obtain

$$a'^2 = \frac{P^2}{a_0^{\,2}} - \frac{P^2}{a^2} + 2Q \ln \frac{a}{a_0} , \tag{5}$$

where $a_0$ is the value of a at s = 0. Equation (5) can be integrated to obtain

$$s = \int_{a_0}^{a} \frac{da}{\left[\frac{P^2}{a_0^{\,2}} - \frac{P^2}{a^2} + 2Q \ln \frac{a}{a_0}\right]^{\frac{1}{2}}} . \tag{6}$$

One therefore has

$$\frac{L}{2} = \frac{a_0^{\,2}}{P} \int_{1}^{a_1/a_0} dz \left[1 - \frac{1}{z^2} + \epsilon \ln z\right]^{-\frac{1}{2}} , \tag{7}$$

where $a_1$ is the value of a at the lens and where

$$\epsilon = \frac{2Qa_0^2}{P^2} . \tag{8}$$

A properly matched beam will have

$$\frac{a_1'}{a_1} = \frac{1}{2f} . \tag{9}$$

This gives

$$2f = \frac{a_0^2}{P} z_1 \left[ 1 - \frac{1}{z_1^2} + \epsilon \ln z_1 \right]^{-\frac{1}{2}} , \tag{10}$$

where

$$z_1 = a_1/a_0. \tag{11}$$

One can think of $a_0$ and $a_1$ as the independent parameters, with L and f then being obtained from Eqs. (7) and (10).

It is also useful to calculate the phase advance per period for each point in transverse phase space. This can be shown to be

$$\frac{\mu}{2} = P \int_0^{L/2} \frac{ds}{[a(s)]^2} = \int_1^{a_1/a_0} \frac{dz}{z^2} \left[ 1 - \frac{1}{z^2} + \epsilon \ln z \right]^{-\frac{1}{2}} \tag{12}$$

We can obtain simple approximations valid near $a_1/a_0 = 1$. Setting

$$\frac{a_1}{a_0} = 1 + \Delta^2 , \tag{13}$$

we have

$$L \simeq \frac{4}{\sqrt{2+\epsilon}} \frac{a_0^2}{P} \Delta , \tag{14a}$$

$$f \simeq \frac{1}{2\sqrt{2+\epsilon}} \frac{a_0^2}{P} \frac{1}{\Delta} , \tag{14b}$$

$$\mu \simeq \frac{4}{\sqrt{2+\epsilon}} \Delta . \tag{14c}$$

For $\epsilon >> 1$, these become

$$L \simeq \frac{4a_0 \Delta}{\sqrt{2Q}} , \tag{15a}$$

$$f \simeq \frac{a_0}{2\Delta\sqrt{2Q}} , \tag{15b}$$

$$\mu \simeq \frac{4P\Delta}{a_0\sqrt{2Q}} . \tag{15c}$$

A further useful approximation is the relation between quadrupole strength, magnet length, and equivalent thin lens focal length for a triplet. This is, for an $\ell$, $\ell_1$, $2\ell$, $\ell_1$, $\ell$ triplet

$$\frac{1}{f} \simeq \frac{4}{3} K_0^2 \ell^3 + 2K_0^2 \ell^2 \ell_1 , \tag{16}$$

where $K_0$ is defined as the coefficient of x in the equation

$$x'' + K_0 x = 0 \tag{17a}$$

$$y'' - K_0 y = 0 , \tag{17b}$$

within a magnet, and $\ell_1$ is the effective separation between the ends of the quadrupole elements which are of length $\ell$, $2\ell$, $\ell$.

## III. Fringing Field Effects

An approximate form for the fringing impulse can be derived by assuming a field on axis with sharp boundaries.[2] The result for a triplet of length $\ell$, $2\ell$, $\ell$ and "gradient" $-K_0$, $K_0$, $-K_0$ is

$$\Delta x' \simeq -K_0^2 \ell (3xy^2 + x^3), \qquad \Delta x = 0 \tag{18a}$$

$$\Delta y' \simeq -K_0^2 \ell (3x^2 y + y^3), \qquad \Delta y = 0 \tag{18b}$$

If we assume an ellipsoidal shape in the 4-dimensional phase space, the projection in x, x' space will be determined by those particles with no motion in the y direction. The distortion of this phase projection will therefore be determined by the $x^3$ and $y^3$ terms in Eq. (18a) and (18b) respectively.

The $xy^2$ and $x^2y$ terms in Eqs. (18a) and (18b) will lead to an increase in $r^2_{max} = x^2_{max} + y^2_{max}$ which will not show up as a distortion of the boundary in either the xx' or yy' projections. The two effects will therefore be evaluated separately.

### A. Non-Linear Uncoupled Terms

The particles can now be considered to have displacements and angles at the triplets given by

$$\begin{aligned} x &= A \sin\phi & \qquad y &= B \sin\psi \\ x' &= \frac{A}{\beta_t} \cos\phi & \qquad y' &= \frac{B}{\beta_t} \cos\psi \end{aligned} \tag{19}$$

where $\beta_t$ is the Courant-Snyder $\beta$ at the triplet. In terms of the envelope

$$\beta_t = \frac{a_1^2}{P} \tag{20}$$

and $A = a_1$ for the unperturbed beam. The change in amplitude due to the fringing field impulse is therefore given by

$$\frac{\Delta A}{A} = \frac{-\beta_t^2 x' \Delta x'}{A^2} = -K_0^2 \ell \beta_t A^2 \sin^3 \phi \cos \phi$$
$$= -\frac{K_0^2 \ell a_1^4}{8P} \left[ 2 \sin 2\phi - \sin 4\phi \right] \tag{21}$$

It is necessary to sum Eq. (21) over all triplets. This is accomplished by writing $\phi = \phi_0 + n\mu$ and summing over n. The distortion then separates into a part depending on $2\phi_0$ which corresponds to the normal elliptical transformation of a linear element, and a $4\phi_0$ part which can only be interpreted as an equivalent beam growth. The maximum values of these equivalent distortions are given, for small $\mu$, by

$$\frac{\Delta A_2}{A} \sim \frac{K_0^2 \ell a_1^4}{4P\mu} \quad \text{"elliptical"} \tag{22a}$$

$$\frac{\Delta A_4}{A_4} \sim \frac{K_0^2 \ell a_1^4}{16 P \mu} \quad \text{"spherical"} \tag{22b}$$

valid for either the x or y oscillation. If we consider a triplet with $\ell_1 = 0$, we can express Eq. (22) in terms of the parameters $\epsilon$, $a_0$, P, $\Delta$ using Eqs. (13), (14), and (16). This leads to

$$\frac{\Delta A_2}{A} \sim \frac{3(2+\epsilon)}{32} \frac{a_0^2}{\ell^2} (1+\Delta^2)^4 \,, \tag{23a}$$

$$\frac{\Delta A_4}{A} \sim \frac{3(2+\epsilon)}{128} \frac{a_0^2}{\ell^2} (1+\Delta^2)^4 \,. \tag{23b}$$

Although all expressions are valid only to the lowest order in $\Delta$, the $(1+\Delta^2)^4$ is retained since Eq. (23) is strongly dependent on $\Delta$.

B. Coupled Terms

For the coupled terms one similarly has

$$A\Delta A \simeq -\frac{3}{8} \frac{K_0 \ell a_1^2 A^2 B^2}{P} [2 \sin 2\phi - \sin 2(\phi-\psi) - \sin 2(\phi+\psi)] \tag{24a}$$

$$B\Delta B \simeq -\frac{3}{8} \frac{K_0^2 \ell a_1^2 A^2 B^2}{P} [2 \sin 2\psi + \sin 2(\phi-\psi) - \sin 2(\phi+\psi)] \tag{24b}$$

The $\phi-\psi$ terms can accumulate over several triplets, but these fortunately lead to no change in $R^2 = A^2 + B^2$. In fact one finds a maximum effect for $A^2 = B^2 = a_1^2/2$ giving

$$\frac{\Delta R}{R} \simeq -\frac{3}{16} \frac{K_0^2 \ell a_1^4}{P} [\sin 2\phi + \sin 2\psi - \sin 2(\phi+\psi)] \,. \tag{24c}$$

Once again the distortion separates into an elliptical part ($2\phi$ or $2\psi$) which may be removable by a linear element, and a 4th harmonic part which represents equivalent beam growth. Specifically, for small $\mu$.

$$\frac{\Delta R_2}{R} \sim \frac{3K_0^2 \ell a_1^4}{8P\mu} \tag{25a}$$

$$\frac{\Delta R_4}{R} \sim \frac{3K_0^2 \ell a_1^4}{32 P \mu} \tag{25b}$$

For the triplet with $\ell_1 = 0$, we can write

$$\frac{\Delta R_2}{R} \sim \frac{9(2+\epsilon)}{64} \frac{a_0^2}{\ell^2} (1+\Delta^2)^4 \tag{26a}$$

$$\frac{\Delta R_4}{R} \sim \frac{9(2+\epsilon)}{256} \frac{a_0^2}{\ell^2} (1+\Delta^2)^4 \tag{26b}$$

C. Compensation of "Elliptical" Term

The form of the $2\phi$ term in Eqs. (21) and (24a) suggests that one can adjust the focal length of the triplet in order to provide some compensation. If the focal strength of each triplet is reduced by

$$\frac{1}{f_{eq}} = \frac{1}{f} - \delta , \tag{27}$$

there will be an additional (small impulse) given by

$$\Delta x' \simeq x\delta , \tag{28a}$$

$$\Delta y' \simeq y\delta , \tag{28b}$$

leading to the amplitude changes.

$$A\Delta A \simeq \frac{A^2 a_1^2 \delta}{2P} \sin 2\phi , \tag{29a}$$

$$B\Delta B \simeq \frac{B^2 a_1^2 \delta}{2P} \sin 2\psi . \tag{29b}$$

As a result Eq. (22a) is changed to

$$\frac{\Delta A_2}{A} \sim \frac{a_1^{\;2}}{4P\mu} \left| K_0^{\;2} \ell a_1^{\;2} - 2\delta \right| , \tag{30}$$

and Eq. (25a) is changed to

$$\frac{\Delta R_2}{R} \sim \frac{3a_1^{\;2}}{8P\mu} \left| K_0^{\;2} \ell a_1^{\;2} - \frac{4}{3}\delta \right| . \tag{31}$$

A reasonable choice for $\delta$ is

$$\delta = \frac{3}{5} K_0^{\;2} \ell a_1^{\;2} , \tag{32}$$

which gives

$$\frac{\Delta A_2}{A} \sim \frac{K_0^{\;2} \ell a_1^{\;4}}{20P\mu} , \tag{33}$$

and

$$\frac{\Delta R_2}{R} \sim \frac{3K_0^{\;2} \ell a_1^{\;4}}{40P\mu} , \tag{34}$$

leaving the two effects with the same relative value as Eqs. (22b) and (25b). If the compensated results in Eqs. (33) and (34) are then combined with (22b) and (25b) we have as an upper limit for the apparent increase in beam size

$$\frac{\Delta A}{A} \sim \frac{\Delta B}{B} \sim \frac{9K_0^{\;2} \ell a_1^{\;4}}{80P\mu} \simeq \frac{27(2+\epsilon)}{40} \frac{a_0^{\;2}}{(4\ell)^2} (1 + \Delta^2)^4 \tag{35}$$

$$\frac{\Delta R}{R} \sim \frac{27\, K_0^2\, \ell\, a_1^4}{160\, P\, \mu} \simeq \frac{81(2+\epsilon)}{80} \frac{a_0^2}{(4\ell)^2} (1+\Delta^2)^4 \tag{36}$$

## IV. Additional Coupling Effects

Similar distortions can be caused by higher multipole components of the main quadrupole field. In fact the similarity of the impulses due to a $2^4$-pole component to those in Eqs. (18a) and (18b) suggest that the relative importance of the non-linear term to the coupling term can be changed if desired.

If the field in the central member of each triplet is derived from the scalar potential

$$\Psi = \text{const}\ [K_0 x y - K_1 (x^3 y - x y^3)], \tag{37}$$

then Eqs. (17a) and (17b) become

$$x'' + K_0 x = -K_1 (3xy^2 - x^3) \quad , \tag{38a}$$

$$y'' - K_0 y = -K_1 (3x^2 y - y^3) \quad , \tag{38b}$$

corresponding to the additional impulse

$$\Delta x' \simeq 2K_1 \ell\, (3xy^2 - x^3) \quad , \tag{39a}$$

$$\Delta y' \simeq 2K_1 \ell\, (3x^2 y - y^3) \quad . \tag{39b}$$

It is clear that the nonlinear $(x^3, y^3)$ and coupling $(x^2 y, xy^2)$ terms in Eq. (18) can be exchanged for each other by choosing $K_1$ appropriately.

For example, if $2K_1 = K_0^2$, the borders of the x-x' and y-y' projections will not grow. Of course the radius of the beam in a 45 deg direction ($x = \pm y$) will increase by a correspondingly larger amount. It should be pointed out however that any condition like $2K_1 = K_0^2$ implies compensation depending on magnetic excitation. This would require modification of the usual quadrupole designs, including additional windings, if the compensation is to be correct for different gradient settings.

## V. Summary

We have made a crude estimate of the distortion produced by quadrupole fringing fields for a periodic transport system. This estimate contained in Eqs. (35) and (36), indicates that one should design a system with $a_0$ and $\Delta$ as small as possible and $\ell$ as large as possible. For transport of high current proton beams in the region between the electrostatic accelerator and the linac, the parameter $\epsilon$ is generally quite large compared to 1. In this case the combination of parameters in Eqs. (35) and (36), governing the magnitude of the distortion, is

$$\frac{I}{\beta^3} \frac{a_1^4}{P^2 \ell^2} \ . \tag{39}$$

Of course, there is a practical lower limit to $a_1$ since the smaller $a_1$ is, the closer together the triplets must be.

One point which should be mentioned is that the transport system may consist of equally spaced quadrupole magnets instead of triplets, much as within the linac. This should permit lower quadrupole fields and less distortion in the fringing fields. However our previous analysis is made difficult because the beam no longer has circular symmetry.

Any actual design will also have to include matching both to the emittance of the source and to the admittance of the linac. Moreover, the beam will become bunched following the buncher region, and this will change the magnitude of the space charge force. For this and many other reasons the results in this note are only intended to be guides, and actual orbit calculations through real transport systems will be necessary to establish the relevant parameters.

The present work also indicates that the major part of the quadrupole fringing field distortion can be removed by small adjustments in the quadrupole strengths. The remaining distortion consists of a growth both in the $x$-$x'$, $y$-$y'$ projections, and in the beam radius. If desirable, one can be reduced at the expense of the other by building higher multipoles into the quadrupole fields.

## FOOTNOTE AND REFERENCES

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[1] E. Regenstreif, Proceedings of the Los Alamos Linac Conference, October 1966, p. 245; Proceedings of the Brookhaven Linac Conference, May 1968; P. W. Allison, Los Alamos Internal Report MP-4/PWA-1, July 27, 1967; P. W. Allison and R. R. Stevens, Proceedings of the Los Alamos Linac Conference, May 1968.

[2] R. L. Gluckstern, Proceedings of the Los Alamos Linac Conference, October 1966, p. 250; Gluckstern, Stevens, and Allison, Los Alamos Internal Report MP-DO/2, August 1967.